M-math 2nd year Mid Semester Exam
Subject : Probability Theory

Time : 3.00 hours   Max.Marks 50.

1. Let $\{X_n; n \geq 0\}$ be the simple symmetric random walk on $\mathbb{Z}$ starting at zero. For $x \in \mathbb{Z}$, let $\tau_x := \inf\{n : X_n = x\}$. For $a < 0 < b$, show that

$$P\{\tau_a < \tau_b\} = \frac{b}{b-a}.$$

(10)

2. Let $\{X_n; n \geq 0\}$ be a square integrable martingale in its natural filtration, with square variation process $\{< X >_n; n \geq 0\}$. Let τ be a finite stopping time (in the natural filtration of X) such that $E < X >_\tau \; < \infty$. Show that

$$E(X_\tau - X_0)^2 = E < X >_\tau \quad \text{and} \quad EX_\tau = EX_0.$$

(15)

3.Let $f : [0,1) \to \mathbb{R}$ be an integrable function with respect to Lebesgue measure λ . Let $I_{n,k} := [k2^{-n}, (k+1)2^{-n})$ for $n \in \mathbb{N}$ and $k = 0, 1, \cdots, 2^n - 1$. Define $f_n : [0,1) \to \mathbb{R}$ as follows : if $x \in I_{k,n}$,

$$f_n(x) := 2^n \int_{I_{k,n}} f \, d\lambda.$$

Show that $\{f_n; n \geq 1\}$ is a uniformly integrable martingale in an appropriate filtration and deduce that $f_n(x) \to f(x)$ for λ almost all $x \in [0,1)$. (15)

4. Let $\{X_i; i \geq 0\}$ be a sequence of i.i.d random variables and let N be a Poisson r.v. independent of $\{X_i; i \geq 0\}$. Let $Y := X_0 + X_1 + \cdots + X_N$. Show that Y is an infinitely divisible random variable. (10)

5. Let Y be a Binomial random variable with parameters $n \geq 1$ and p. $0 < p < 1$. Is Y infinitely divisible ? Justify your answer. (10)

6. Let $0 < \alpha < 2$, and define a measure ν_α on $\mathbb{R}$ by $\nu_\alpha(dx) := \theta_\alpha^{-1}|x|^{-\alpha-1}dx$, where θ_α is a positive constant. Show that ν_α is a canonical measure and

hence that there exists an infinitely divisible probability measure μ_α on $\mathbb{R}$ with the canonical triple $(0, 0, \nu_\alpha)$. Show that for a suitable choice of $\theta_\alpha, \psi_\alpha(t)$, the logarithm of the characteristic function of μ_α is given as $\psi_\alpha(t) = -|t|^\alpha$. (10)